$K-$structure of the $U(\mathfrak{g})^K-$module $U(\mathfrak{g})$ for simple Lie algebras $\mathfrak{g} = \mathfrak{su}(n, 1)$ and $\mathfrak{g} = \mathfrak{so}(n, 1)$

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Let $\mathfrak{g}$ be a simple real Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition, G the adjoint group of $\mathfrak{g}$, K its maximal compact subgroup with Lie algebra $\mathfrak{k}$. Further, denote by $U(\mathfrak{g})$ and $U(\mathfrak{k}) \subseteq U(\mathfrak{g})$ the complexified universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{k}$ and let $Z(\mathfrak{g})$ and $Z(\mathfrak{k})$ be its centers. Let $U(\mathfrak{g})^K$ be the subalgebra of $K-$invariants in $U(\mathfrak{g})$. Then obviously we have a morphism of algebras $Z(\mathfrak{g}) \otimes Z(\mathfrak{k}) \longrightarrow U(\mathfrak{g})^K$ defined by the multiplication. Knop has proved that for noncompact $\mathfrak{g}$ this morphism is always injective and that its image is exactly the center of the algebra $U(\mathfrak{g})^K$. Furthermore, the algebra $U(\mathfrak{g})^K$ is commutative, i.e. isomorphic to $Z(\mathfrak{g}) \otimes Z(\mathfrak{k})$, if and only if $\mathfrak{g}$ is either $\mathfrak{su}(n, 1)$ or $\mathfrak{so}(n, 1)$. In these cases $U(\mathfrak{g})$ is free as a $U(\mathfrak{g})^K-$module. We show that in these cases the multiplication defines an isomorphism of $K-$modules and $U(\mathfrak{g})^K-$modules $U(\mathfrak{g})^K \otimes H \longrightarrow U(\mathfrak{g})$, where H is the subspace of $U(\mathfrak{g})$ spanned by all powers x^k, $k \in \mathbb{Z}_+$, $x \in \mathcal{N}_K$, and $\mathcal{N}_K$ is the variety of all nilpotent elements in $\mathfrak{g}^{\mathbb{C}}$ whose projection to $\mathfrak{k}^{\mathbb{C}}$ along $\mathfrak{p}^{\mathbb{C}}$ is nilpotent in the reductive Lie algebra $\mathfrak{k}^{\mathbb{C}}$. Furthermore, we study the structure of the $K-$module H and show that the multiplicity of every irreducible representation δ of K in it equals its dimension $d(\delta)$. In other words, as a $K-$module H is equivalent to the regular representation of K. A simple consequence is that for any finitedimensional $K-$module V the space $(U(\mathfrak{g}) \otimes V)^K$ is a free $U(\mathfrak{g})^K-$module of rank dim V.