On the Extension of a $D(4)$-triple $\{1, b, c\}$

Workshop on Diophantine m-tuples and Modular Curves

Sanda Bujačić Babić
(joint work with Alan Filipin and Ana Jurasić)

Faculty of Mathematics, University of Rijeka

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Introduction

- The concept of Diophantine m-tuples dates back to ancient Greek mathematics.
- Diophantus of Alexandria[1] was exploring sets of rational numbers that satisfy specific properties.
- The classification and extension of such sets have remained one of the central focuses in number theory for centuries and have yielded meaningful insights across different domains.

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[1] A prominent mathematician active in the 3rd century.

Basic Definitions

Definition 1 (Diophantine m-tuple with property $D(n)$)

Let $n \in \mathbb{Z} \backslash \{0\}$. A set of m positive integers $\{a_1, a_2, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$[2] if

$$a_i a_j + n$$

is a perfect square for all $1 \leq i < j \leq m$.

A Diophantine m-tuple with the property $D(1)$ is simply called a Diophantine m-tuple.

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[2] or simply a $D(n)$-m-tuple.

Historical Examples

The earliest known **rational** Diophantine quadruple was found by Diophantus:

$$\left\{ \frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16} \right\}.$$

The first Diophantine quadruple consisting of four **positive integers**[3] was discovered by Pierre de Fermat:

$$\{1, 3, 8, 120\}.$$

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[3] This set is widely known as *Fermat's set*.

The Classical Case: $n = 1$

Historically, the case $n = 1$ has been investigated most thoroughly. A few breakthrough results include:

- **Baker & Davenport (1969):** Proved that Fermat's set cannot be extended by a positive integer to a Diophantine quintuple [2].
- **Dujella (2004):** Established that there are no Diophantine sextuples and that there exist only finitely many Diophantine quintuples over the integers [5].
- **He, Togbé & Ziegler (2019):** Solved a long-standing open problem by proving the non-existence of Diophantine quintuples [7].

Different Directions of Generalization

Research has expanded the study in several fundamental directions:

1. Investigating $D(n)$-m-tuples for a fixed integer $n \neq 1$ (such as $n = -1$ or $n = 4$) within the ring of integers $\mathbb{Z}$.
2. Extending the problem from the ring of integers $\mathbb{Z}$ to the field of rational numbers $\mathbb{Q}$ or other algebraic number fields.
3. Investigating polynomial Diophantine tuples over rings such as $\mathbb{Z}[X]$, $\mathbb{Q}[X]$, or $\mathbb{C}[X]$.

A Generalization of the Problem: $n = 4$

A highly notable and intensely studied case is when $n = 4$:

- **Filipin (2008, 2011)** proved the non-existence of $D(4)$-sextuples and showed that there are only finitely many $D(4)$-quintuples [8, 9].
- **Bliznac Trebješanin & Filipin (2019)** proved the non-existence of $D(4)$-quintuples [4].

The Extension of a $D(4)$-triple $\{1, \text{b}, \text{c}\}$

Let $\{1, b, c\}$ be a $D(4)$-triple such that $1 < b < c$. In this case,

$$b + 4 = r^2, \quad c + 4 = s^2, \quad bc + 4 = t^2,$$

$r, s, t \in \mathbb{Z}$.

- We address the problem of extending the $D(4)$-triple $\{1, b, c\}$, where $1 < b < c$, by adding a larger element.
- We combine several previous results on $D(4)$-quadruples, the classical method based on solving binary recurrence sequences, and some new approaches.
- Our aim is to determine whether there exists an irregular $D(4)$-quadruple for certain values of c.

The Extension of a $D(4)$-triple {1, b, c}

Generally, the $D(4)$-triple $\{a, b, c\}$, where $a < b < c$, can always be extended by element $d_{\pm}$ of the form

$$d_{\pm} = a + b + c + \frac{1}{2}(abc \pm \sqrt{(ab+4)(ac+4)(bc+4)}). \qquad (1)$$

It is easy to check that $\{a, b, c, d_+\}$ and $\{a, b, c, d_-\}$, if $d_- \neq 0$, are $D(4)$-quadruples, where $d_- < c$.

Example 2

$$\{1, b, d_-, c\} = \{1, 5, 12, 96\}$$

$$\{1, b, c, d_+\} = \{1, 5, 96, 672\}.$$

The Extension of a $D(4)$-triple {1, b, c}

Definition 3

Diophantine $D(4)$-quadruple $\{a, b, c, d\}$ is called regular if

$$(a+b-c-d)^2 = (ab+4)(cd+4),$$

or equivalently if $d = d_\pm$ where

$$d_\pm = a+b+c+\frac{1}{2}(abc \pm rst),$$

and $r, s, t \in \mathbb{Z}$ are defined by

$$ab+4 = r^2, \quad ac+4 = s^2, \quad bc+4 = t^2.$$

Conjecture 1 ([6])

Every $D(4)$-quadruple $\{a, b, c, d\}$, where $a < b < c < d$, is regular i.e. $d = d_+$.

Related Work

- **Adédji**, **Filipin** and **Togbé** [1] studied the extensibility of the $D(4)$-triple $\{1, b, c\}$, where $1 < b < c$, and proved that such set cannot be extended to an irregular $D(4)$-quadruple for some values of c depending on b.

Our aim is to deal with the Conjecture 1 for $a = 1$ i.e. to answer the question:

Is every $D(4)$-quadruple $\{1, b, c, d\}$, where $1 < b < c < d$, regular?

Extending a $D(4)$-Pair to a $D(4)$-Triple

Assume we have a $D(4)$-pair $\{1, b\}$ such that $1 < b$, where:

$$b + 4 = r^2, \quad r \in \mathbb{Z}. \tag{2}$$

We are interested in finding integers c that extend this pair to a $D(4)$-triple $\{1, b, c\}$ with $1 < b < c$. Alongside (2), the following conditions must hold:

$$c + 4 = s^2, \quad bc + 4 = t^2, \quad s, t \in \mathbb{Z}. \tag{3}$$

Lemma 4 ([4], Lemma 1)

Let $\{1, b, c\}$, where $1 < b < c$, be a $D(4)$-triple. Then

$$c = 1 + b + 2r \quad \text{or} \quad c > 4b.$$

Extending a $D(4)$-Pair to a $D(4)$-Triple

- The smallest such c is $c = 1 + b + 2r$, for which we obtain a triple referred to a **regular triple**.
- Beside that c, there are infinitely many extensions of the pair to the triple each corresponding to a solution (t, s) of the Pellian equation
$$t^2 - bs^2 = 4(1 - b), \tag{4}$$
obtained from (3) by eliminating c.

Extending a $D(4)$-Pair to a $D(4)$-Triple

According to [4, Lemma 8], every positive integer solution (t, s) of the Pellian equation (4) can be generated via the relation:

$$t + s\sqrt{b} = (t_0 + s_0\sqrt{b})\left(\frac{r + \sqrt{b}}{2}\right)^n, \quad n \in \mathbb{N}_0 \tag{5}$$

Here, (t_0, s_0) represents a fundamental solution of (4) that satisfies bounds:

$$2 \leq s_0 \leq \sqrt{\frac{r^2 - 5}{r - 2}} \quad \text{and} \quad 2 \leq |t_0| \leq \sqrt{(r - 2)(r^2 - 5)}. \tag{6}$$

Extending a $D(4)$-Pair to a $D(4)$-Triple

- By [11, Theorem 108a], the Pellian equation (4) has only **finitely many** fundamental solutions (t_0, s_0).

- Each fundamental solution generates (t_n, s_n) by:

$$t_n + s_n\sqrt{b} = (t_0 + s_0\sqrt{b})(T_n + U_n\sqrt{b}), \quad n \geq 0. \tag{7}$$

Sequences $(T_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ satisfy the linear recurrence relations:

$$T_0 = 1, \quad T_1 = \frac{r}{2}, \quad T_{n+2} = rT_{n+1} - T_n, \tag{8}$$

$$U_0 = 0, \quad U_1 = \frac{1}{2}, \quad U_{n+2} = rU_{n+1} - U_n. \tag{9}$$

Extending a $D(4)$-Pair to a $D(4)$-Triple

We easily get:

$$(t, s) = (t_n, s_n) = (T_n t_0 + U_n s_0 b,\ T_n s_0 + U_n t_0). \tag{10}$$

The sequence of elements:

$$c_n = s_n^2 - 4, \tag{11}$$

leads to the infinite family of $D(4)$-triples:

$$\{1, b, c_n\}, \quad n \in \mathbb{N}. \tag{12}$$

The Case $(t_0, s_0) = (\pm 2, 2)$

For the fundamental solution $(t_0, s_0) = (\pm 2, 2)$, Adédji, Filipin, and Togbé [1] obtained 8 possibilities for c that allow further extensions to a $D(4)$-quadruple $\{1, b, c, d\}$ with $d > d_+$:

$$\begin{aligned}
c_1^{\pm} &= r^2 \pm 2r - 3,\\
c_2^{\pm} &= r^4 \pm 2r^3 - 3r^2 \mp 4r,\\
c_3^{\pm} &= r^6 \pm 2r^5 - 5r^4 \mp 8r^3 + 7r^2 \pm 6r - 3,\\
c_4^{\pm} &= r^8 \pm 2r^7 - 7r^6 \mp 12r^5 + 16r^4 \pm 20r^3 - 12r^2 \mp 8r.
\end{aligned}$$

The Fundamental Solution and Regularity

- Since $(t_0, s_0) = (\pm 2, 2)$ is a fundamental solution of the equation (4) for every r, this way we obtain $D(4)$-triples $\{1, b, c_n^{\pm}\}$ for every $D(4)$-pair $\{1, b\}$, where $b = r^2 - 4$ and $r \geq 3$.
- Since $b + 4 = r^2$ and $c_1^{+} + 4 = (r+1)^2$, $D(4)$-triples of the form $\{1, b, c_1^{+}\}$, for $r \geq 3$, have the smallest possible gap between b and c i.e. they are regular triples.

The Case $r = 10$: The Pair $\{1, 96\}$

- For $r \leq 9$, the only fundamental solution of (4) is $(t_0, s_0) = (\pm 2, 2)$.
- For $r = 10$ ($b = 96$), an additional fundamental solution appears:
$$(t_0, s_0) = (\pm 22, 3).$$

In this specific case, we get:

$$(t_n^{\pm}, s_n^{\pm}) = (\pm 22 T_n + 288 U_n,\ 3 T_n \pm 22 U_n).$$

Consequently, the $D(4)$-pair $\{1, 96\}$ can be extended to new $D(4)$-triples where:

$$\begin{aligned} c_n^{-} &= 12,\ 1365,\ \ldots \\ c_n^{+} &= 672,\ 66045,\ \ldots \end{aligned}$$

Experimental Results for $r \leq 100$

A systematic computer search for $r \leq 100$ revealed an interesting pattern: For every r of the form

$$r = g(g-3), \; g \geq 5,$$

there is an additional fundamental solution of (4) of the form

$$(t_0, s_0) = (\pm(g^3 - 5g^2 + 4g + 2), g - 2).$$

For r of this form, s_0 is the biggest possible.

$$\begin{aligned} g = 5 &\implies r = 10: \quad (t_0, s_0) = (\pm 22, 3) \\ g = 6 &\implies r = 18: \quad (t_0, s_0) = (\pm 62, 4) \\ g = 7 &\implies r = 28: \quad (t_0, s_0) = (\pm 128, 5) \; \ldots \end{aligned}$$

$r = g(g-3),\ g \geq 5$

The sequences $(T_n)_{n\geq 0}$ and $(U_n)_{n\geq 0}$ satisfy linear recurrence relations:

$$T_0 = 1, \quad T_1 = \frac{g(g-3)}{2}, \quad T_{n+2} = g(g-3)T_{n+1} - T_n$$
$$U_0 = 0, \quad U_1 = \frac{1}{2}, \quad U_{n+2} = g(g-3)U_{n+1} - U_n$$

The infinite family of triples is obtained via $c_n = s_n^2 - 4$, starting from $c_0 = g^2 - 4g$.

Analytical Bounds for Further Extensions

- By [3, Theorem 1.6], a $D(4)$-triple $\{1, b, c_n\}$ cannot be extended to a $D(4)$-quadruple $\{1, b, c_n, d\}$ with

  $$1 < b < c_n = c < d_+ < d$$

  if $c \geq 39247b^4$. Therefore, to search for further extensions, we must assume:

  $$c < 39247b^4. \tag{13}$$

- For such quadruples, Blizanac Trebješanin [3] established strict lower bounds:

  $$b > 10^5 \tag{14}$$

  $$d > 0.249965 \cdot 10^{27.5} \cdot c^{6.5} \tag{15}$$

The Classical Approach

We consider a general problem of extending a $D(4)$-triple $\{1, b, c\}$ to a $D(4)$-quadruple $\{1, b, c, d\}$, where $d > c$. This requires solving the system of simultaneous equations

$$d + 4 = x^2, \quad bd + 4 = y^2, \quad cd + 4 = z^2 \tag{16}$$

where d, x, y and z are positive integers.
There exist regular quadruples $\{1, b, c, d_{\pm}\}$, where $d_{\pm}$ is defined by (1) and $d_{-} \neq 0$. Additionally, $d_{-} < c$ and $d_{+} > c$.

Motivation and Objective

Known Results (Adédji, Filipin, Togbé [1])

For the fundamental solution $(t_0, s_0) = (\pm 2, 2)$, it is proven that every $D(4)$-quadruple $\{1, b, c_n^{\pm}, d\}$ with

$$1 < b < c = c_n^{\pm} < d_+ < d$$

is **regular**.

Scope of the Research: $r = g(g-3)$

Our goal is to determine whether every extension of the $D(4)$-triple $\{1, b, c\}$ parameterized by $r = g(g-3)$, $g \geq 5$ to a quadruple $\{1, b, c, d\}$ is necessarily regular.

The Classical Approach

For $d = d_\pm$, we have

$$\begin{aligned} d_\pm + 4 &= u_\pm^2, \\ bd_\pm + 4 &= v_\pm^2, \\ cd_\pm + 4 &= w_\pm^2, \end{aligned} \tag{17}$$

where

$$u_\pm = \frac{t \pm rs}{2}, \quad v_\pm = \frac{bs \pm rt}{2}, \quad w_\pm = \frac{cr \pm st}{2}.$$

We can easily get

$$t < rs, \; bs < rt, \quad \text{i.e.} \quad u_- < 0, \quad v_- < 0 \quad \text{and} \quad w_- > 0.$$

The Classical Approach: The System of Pellian Equations

In our observations, we will assume that $\{1, b, c, d\}$ is an irregular $D(4)$-quadruple and we will examine if there exist such $d > c$. Eliminating d from (16), the following system of Pellian equations arise

$$\begin{aligned} z^2 - cx^2 &= 4(1-c), & (18)\\ bz^2 - cy^2 &= 4(b-c), & (19)\\ y^2 - bx^2 &= 4(1-b). & (20) \end{aligned}$$

Each of equations (18)-(20) has finitely many fundamental solutions (z_0, x_0), (z_1, y_1) and (y_2, x_2), respectively.

The Classical Approach: Binary recurrent sequences

By [5, Lemma 1], from these solutions we can generate all solutions of the equations (18)-(20) with

$$
\begin{aligned}
z + x\sqrt{c} &= (z_0 + x_0\sqrt{c})\left(\frac{s+\sqrt{c}}{2}\right)^m, \quad m \geq 0, && (21)\\
z\sqrt{b} + y\sqrt{c} &= (z_1\sqrt{b} + y_1\sqrt{c})\left(\frac{t+\sqrt{bc}}{2}\right)^n, \quad n \geq 0, && (22)\\
y + x\sqrt{b} &= (y_2 + x_2\sqrt{b})\left(\frac{r+\sqrt{b}}{2}\right)^l, \quad l \geq 0. && (23)
\end{aligned}
$$

The Classical Approach: Binary recurrent sequences for z

Firstly, for any solution (x, y, z) of the system (18)-(20), we have

$$z = v_m = w_n, \quad m, \ n \geq 0, \tag{24}$$

where the sequences $(v_m)_{m\geq 0}$ and $(w_n)_{n\geq 0}$ are obtained using (21) and (22) and given by

$$v_0 = z_0, \quad v_1 = \frac{1}{2}(sz_0 + cx_0), \quad v_{m+2} = sv_{m+1} - v_m, \tag{25}$$

$$w_0 = z_1, \quad w_1 = \frac{1}{2}(tz_1 + cy_1), \quad w_{n+2} = tw_{n+1} - w_n. \tag{26}$$

The Classical Approach: Binary Recurrent Sequences for y

Similarly, for any solution (x, y, z) of the system (18)–(20), we get:

$$y = A_n = B_l, \quad n, l \geq 0. \tag{27}$$

The sequences $(A_n)_{n\geq 0}$ and $(B_l)_{l\geq 0}$ are defined recursively by:

$$A_0 = y_1, \quad A_1 = \frac{1}{2}(ty_1 + bz_1), \quad A_{n+2} = tA_{n+1} - A_n \tag{28}$$

$$B_0 = y_2, \quad B_1 = \frac{1}{2}(ry_2 + bx_2), \quad B_{l+2} = rB_{l+1} - B_l \tag{29}$$

The Classical Approach: Methodological Challenges

While we follow the classical framework of reducing the system of Pellian equations to intersections of binary recurrent sequences, our analysis introduces significant complications, requiring the examination of multiple additional subcases.

Lemma 5

Suppose that $\{1, b, c, d\}$ *is a* $D(4)$*-quadruple such that* $1 < b < c < d_+ < d$, *and that* $(v_m)_{m\geq 0}$ *and* $(w_n)_{n\geq 0}$ *are defined with (25) and (26), respectively.*

1) If $v_{2m} = w_{2n}$ *has a solution, then*

  a) $z_0 = z_1 = \pm 2$, $x_0 = y_1 = 2$ *or*
  b) $z_0 = z_1 = \pm\frac{1}{2}(cr - st)$, $x_0 = \frac{1}{2}(rs - t)$ *and* $y_1 = \frac{1}{2}(rt - bs)$.

2) If $v_{2m} = w_{2n+1}$ *has a solution, then* $z_1 = \pm s$, $z_0 = \mp\frac{1}{2}(cr - st)$, $x_0 = \frac{1}{2}(rs - t)$ *and* $y_1 = r$.

3) If $v_{2m+1} = w_{2n+1}$ *has a solution, then* $z_0 = \pm t$, $z_1 = \pm s$, *and* $x_0 = y_1 = r$.

Lemma 6

Suppose that $\{1, b, c, d\}$ *is a* $D(4)$*-quadruple such that* $1 < b < c < d_+ < d$*, defined by the parametric family* $r = g(g-3)$ *with* $s = g-2$*. Let* $(v_m)_{m\geq 0}$ *and* $(w_n)_{n\geq 0}$ *be defined as before.*

1) *If* $v_{2m} = w_{2n}$ *has a solution, then*
   a) $z_0 = z_1 = \pm 2$, $x_0 = y_1 = 2$ *or*
   b) $z_0 = z_1 = \pm(-g^2+3g+2)$, $x_0 = g-1$ *and* $y_1 = -g^3+4g^2-g-4$.

2) *If* $v_{2m} = w_{2n+1}$ *has a solution, then* $z_1 = \pm(g-2)$, $z_0 = \mp(-g^2+3g+2)$, $x_0 = g-1$ *and* $y_1 = g^2-3g$.

3) *If* $v_{2m+1} = w_{2n+1}$ *has a solution, then* $z_0 = \pm(g^3-5g^2+4g+2)$, $z_1 = \pm(g-2)$, *and* $x_0 = y_1 = g^2-3g$.

Classification of Fundamental Solutions

Lemma 7

Suppose that $\{1, b, c, d\}$ *is a* $D(4)$*-quadruple with* $1 < b < c < d_+ < d$*, and let the sequences* $(A_n)_{n\geq 0}$ *and* $(B_l)_{l\geq 0}$ *be defined by (28) and (29), respectively.*

Then, the fundamental solutions (y_2, x_2) *of equation (20) must be one of the following:*

1) $(y_2, x_2) = (\pm 2, 2)$

2) $(y_2, x_2) = (t_0, s_0)$*, where* (t_0, s_0) *is the fundamental solution of equation (4) from which* c *arises.*

The Classical Approach: Binary Recurrent Sequences for x

Finally, for any solution (x, y, z) of the system (18)–(20), the component x can also be expressed through the intersection of two sequences:

$$x = P_l = Q_m \tag{30}$$

for some non-negative integers l and m.

The sequences $(P_l)_{l\geq 0}$ and $(Q_m)_{m\geq 0}$ are given by the recurrence relations:

$$P_0 = x_2, \quad P_1 = \frac{1}{2}(rx_2 + y_2), \quad P_{l+2} = rP_{l+1} - P_l \tag{31}$$

$$Q_0 = x_0, \quad Q_1 = \frac{1}{2}(sx_0 + z_0), \quad Q_{m+2} = sQ_{m+1} - Q_m \tag{32}$$

System of Binary Recurrence Sequences

- $z = v_m = w_n$:

$$v_{m+2} = (g-2)\,v_{m+1} - v_m$$
$$w_{n+2} = (g^3 - 5g^2 + 4g + 2)\,w_{n+1} - w_n$$

- $y = A_n = B_l$:

$$A_{n+2} = (g^3 - 5g^2 + 4g + 2)\,A_{n+1} - A_n$$
$$B_{l+2} = g(g-3)\,B_{l+1} - B_l$$

- $x = P_l = Q_m$:

$$P_{l+2} = g(g-3)\,P_{l+1} - P_l$$
$$Q_{m+2} = (g-2)\,Q_{m+1} - Q_m$$

By Lemma 5 and Lemma 7, we conclude that these two situations are possible:

- If m and l are even, then:
  a) $z_0 = \pm 2$, $x_0 = 2$, $x_2 = y_2 = 2$ or
  b) $z_0 = \pm 2$, $x_0 = 2$, $x_2 = s_0$, $y_2 = t_0$ or
  c) $|z_0| = \frac{1}{2}(cr - st)$, $x_0 = \frac{1}{2}(rs - t)$, $x_2 = y_2 = 2$ or
  d) $|z_0| = \frac{1}{2}(cr - st)$, $x_0 = \frac{1}{2}(rs - t)$, $x_2 = s_0$, $y_2 = t_0$,

  where (t_0, s_0) is a fundamental solution of (4) from which c arises.
- If m and l are odd, then $z_0 = \pm t$, $x_0 = r$, $x_2 = s_0$, $y_2 = t_0$, where (t_0, s_0) is a fundamental solution of (4) from which c arises.

The cases where m and l have different parity correspond with the above cases.

Technical Remarks: $b^4 < c < 39247b^4$

Remark 1

If $b^4 < c < 39247b^4$, *by [3], the cases* 1.b) *and* 2) *of the Lemma 5 are not possible.*

Also, since $b^4 \geq c > b > 10^5$, *it follows from (15) that:*

$$d > 0.249965 \cdot 10^{157.5}.$$

Technical Remarks: Characterization of d_0 for Small c

Remark 2

In the remaining case where $c \leq b^4$, if the subcase ***1.b)*** *from Lemma 5 occurs, we define*

$$d_0 := \frac{z_1^2 - 4}{c}.$$

Then, by [10], it holds that $d_0 = d_-$ (with $0 < d_0 < c$), meaning $\{1, b, c, d_0\}$ is a ***regular*** *$D(4)$-quadruple.*

Moreover, by [3], this regular element is strictly bounded by:

$$d_- = \frac{z_1^2 - 4}{c} < \frac{\sqrt{c}}{\sqrt{b}} \leq \sqrt[8]{c^3}.$$

Technical Remarks: Characterization of d_0 for Small c

Remark 3

*Similarly, in the case where $c \leq b^4$, if the subcase **2)** from Lemma 5 occurs, we define*

$$d_0 := \frac{z_0^2 - 4}{c}.$$

*Then, by [10], it likewise holds that $d_0 = d_-$ (with $0 < d_0 < c$), meaning $\{1, b, c, d_0\}$ is a **regular** $D(4)$-quadruple.*

Moreover, by [3], this element satisfies the upper bound:

$$d_- = \frac{z_0^2 - 4}{c} < \sqrt{c} \leq b^2.$$

Future Work

- Linear Forms in Logarithms
- Baker-Davenport Reduction

Thank you for attention!

Bibliography I

[1] K. N. Adédji, A. Filipin and A. Togbé, *The problem of the extension of* $D(4)$*-triple* $\{1, b, c\}$, Rad Hrvat. Akad. Znan. Umjet. Mat. Znan., **26** (2022), 21-43.

[2] A. Baker, H. Davenport, The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, Quart. J. Math. Oxford Ser. 20 (1969), 129-137,

[3] M. Blizanac Trebješanin, *Extension of a Diophantine triple with the property* $D(4)$, Acta Math. Hungar. **163** (2021), 213-246.

[4] M. Blizanac Trebješanin and A. Filipin, *Nonexistence of* $D(4)$*-quintuple*, J. Number Theory **194** (2019), 170–217.

[5] A. Dujella, There are only finitely many Diophantine quintuples, J. Reine Angew. Math. 566 (2004), 183-214.

[6] A. Dujella, A. M. S. Ramasamy, *Fibonacci numbers and sets with the property* $D(4)$, Bull. Belg. Math. Soc. Simon Stevin, **12(3)** (2005), 401-412.

Bibliography II

[7] B. He, A. Togbé, V. Ziegler, There is no Diophantine quintuple, Trans. Am. Math. Soc. 371 6665-6709 (2019)

[8] A. Filipin, There does not exist a $D(4)$-sextuple, J. Number Theory 128 (2008), 1555–1565.

[9] A. Filipin, There are only finitely many D(4)-quintuples, Rocky Mountain J. Math. 41 (2011), no. 6, 1847–1859.

[10] A. Filipin, *On the size of sets in which* $xy + 4$ *is always a square*, Rocky Mountain J. Math., **39(4)** (2009), 1195-1224.

[11] T. Nagell, *Introduction to Number Theory*, Almqvist, Stockholm, Wiley, New York, 1951.

[12] N. C. Bonciocat, M. Cipu, M. Mignotte, *There is no Diophantine D(-1)-quadruple*, J. London Math. Soc. **105** (2022), 63-99