Transitive designs constructed from groups

Andrea Švob
(asvob@math.uniri.hr)
Department of Mathematics
University of Rijeka
Croatia

8^{th} April, 2011

Definition

A group G acting on a set Ω is said to be **transitive** *on Ω if it has only one orbit, and so $\alpha^G = \Omega$ for all $\alpha \in \Omega$. Equivalently, G is transitive if for every pair of points $\alpha, \beta \in \Omega$ there exist $x \in G$ such that $\alpha^x = \beta$.*

Definition

If G is a group acting on a set Ω and k is an integer with $1 \leq k \leq |\Omega|$, then we say G is k - **transitive** *if G is transitive on $\Omega^{(k)}$.*

Definition

Let G acts transitively on Ω and let $\alpha \in \Omega$. G is **primitive** *if G_α is a maximal subgroup of G.*

Designs constructed from groups

- 1930, Witt gave a construction of the Witt designs from the Mathieu groups
  - Steiner systems $S(4,5,11), S(5,6,12), S(3,6,22), S(4,7,23), S(5,8,24)$,
  - their automorphism groups are the Mathieu groups, discovered by Mathieu (1861, 1873),
  - the Mathieu groups are the only finite t-transitive permutation groups with $t \geq 4$, except for the symmetric and alternating groups.
- 1993, Haemers, Parker, Pless and Tonchev constructed a $2-(276, 100, 2 \cdot 3^6)$ design, on which Conway's simple group Co_3, as the authomorphism group, acts doubly transitively block - primitive
  - They also constructed the Higman's design $2-(176, 50, 14)$.

Derived design

Given a block B in a symmetric design $2-(v,k,\lambda)$, $\mathcal{D}=(\mathcal{X},\mathcal{B})$, the intersections $B\cap B', B'\in\mathcal{B}, B'\neq B$ form a design $\mathcal{D}'$ with a point set B and parameters $2-(k,\lambda,\lambda-1)$, which is called the **derived design** of $\mathcal{D}$ (with respect to B).

Residual design

The differences $B'-B, B'\in\mathcal{B}, B'\neq B$ form a design $\mathcal{D}''$, with a point set $X-B$ and parameters $2-(v-k,k-\lambda,\lambda)$, which is called the **residual design** of $\mathcal{D}$ (with respect to B).

- 1995, Parker and Tonchev constructed the symmetric $2-(176,50,14)$ design on which the Higman-Sims group acts as a $2-$ transitive authomorphism group
  - $2-(126,36,14)$ - residual design of the Higman design, $U(3,5):Z_2$ acts 2-transitively
  - $2-(50,14,13)$ - derived design of the Higman design, $U(3,5):Z_2$ acts as rank 3 group

- 1999, Held, Hrabe de Angelis and Pavčević: $2-(36,15,6)$ design from the group $PSp_4(3)$,
- 2001, Dempwolff determined the symmetric designs that admit a group $G \leq Aut(G)$, where G is a primitive rank 3 group on points and blocks.

- J.D.Key, J. Moori, Codes, Designs and Graphs from the Janko Groups J_1 and J_2, J. Combin. Math. Combin. Comput. 40 (2002), 143-159.
- **symmetric, $1-$design**

Theorem

Let G be a finite primitive permutation group acting on the set Ω of size n. Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α. If $\mathcal{B} = \{\Delta g : g \in G\}$ and, given $\delta \in \Delta$, $\mathcal{E} = \{\{\alpha, \delta\}g : g \in G\}$, then $\mathcal{D} = (\Omega, \mathcal{B})$ forms a symmetric $1-(n, |\Delta|, |\Delta|)$ design. Further, if Δ is a self-paired orbit of G_α then $\Gamma(\Omega, \mathcal{E})$ is a regular connected graph of valency $|\Delta|$, $\mathcal{D}$ is self-dual, and G acts as an automorphism group on each of these structures, primitive on vertices of the graph, and on points and blocks of the design.

- D. Crnković and V. Mikulić, Unitals, projective planes and other combinatorial structure constructed from the unitary groups $U(3, q), q = 3, 4, 5, 7$, Ars Combin,. to appear

Theorem

Let G be a finite permutation group acting primitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$, $\delta \in \Omega_2$, and let $\Delta_2 = \delta G_\alpha$ be the G_α-orbit of $\delta \in \Omega_2$ and $\Delta_1 = \alpha G_\delta$ be the G_δ-orbit of $\alpha \in \Omega_1$. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta) = (\Omega_2, \mathcal{B})$ is a $1-(n, |\Delta_2|, |\Delta_1|)$ design with m blocks, and G acts as an automorphism group, primitive on points and blocks of the design.

Theorem

Let G be a finite permutation group acting primitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, ..., \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $(\Omega_2, \mathcal{B})$ is a $1 - (n, |\Delta_2|, \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with m blocks, and G acts as an automorphism group, primitive on points and blocks of the design.

Corollary

If a group G acts primitively on the points and the blocks of a $1-$design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem.

- Let G be a simple group and H be a maximal subgroup of G. The conjugacy class of H is denoted by $ccl_G(H)$. Obviously $N_G(H) = H$, so $|ccl_G(H)| = [G : H]$. Denote the elements of $ccl_G(H)$ by $H^{g_1}, H^{g_2}, \ldots, H^{g_j},\ j = [G : H]$.

- Let G be a simple group and H_1 and H_2 be **maximal subgroups** of G. The stabilizer of any element $H_i^x \in ccl_G(H_i),\ i=1,2$, is the maximal subgroup H_i^x, hence G acts primitively on the class $ccl_G(H_i),\ i=1,2$, by conjugation and

$$|ccl_G(H_1)| = [G:H_1] = m,$$

$$|ccl_G(H_2)| = [G:H_2] = n.$$

We can construct a primitive $1-$design such that:

- the point set of the design is $ccl_G(H_2)$,
- the block set is $ccl_G(H_1)$,
- the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i,\ i=1,\ldots,k$, where $\{G_1,...,G_k\} \subset \{H_2^x \cap H_1^y \mid x,y \in G\}$.
- notation: $\mathcal{D}(G,H_2,H_1;G_1,...,G_k)$.

Theorem

Let G be a finite permutation group acting transitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, \ldots, \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and

$$\mathcal{B} = \{\Delta_2 g : g \in G\},$$

then $\mathcal{D}(G, \alpha, \delta_1, \ldots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1-(n, |\Delta_2|, \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with $\frac{m \cdot |G_\alpha|}{|G_{\Delta_2}|}$ blocks. The group $H \cong G / \bigcap_{x \in \Omega_2} G_x$ acts as an automorphism group on $(\Omega_2, \mathcal{B})$, transitive on points and blocks of the design.

Corollary

If a group G acts transitively on the points and the blocks of a $1-$design $\mathcal{D}$, then $\mathcal{D}$ can be obtained as described in Theorem.

We can construct a $1-$design:

- G is a finite group, H_1, H_2, **subgroups** of G
- G acts transitively on the class $ccl_G(H_i)$, $i = 1, 2$, by conjugation and

$$|ccl_G(H_1)| = [G : N_G(H_1)] = m,$$

$$|ccl_G(H_2)| = [G : N_G(H_2)] = n.$$

- Let us denote the elements of $ccl_G(H_1)$ by $H_1^{g_1}, H_1^{g_2}, \ldots, H_1^{g_m}$, and the elements of $ccl_G(H_2)$ by $H_2^{h_1}, H_2^{h_2}, \ldots, H_2^{h_n}$.

We can construct a $1-$design such that:

- the point set of the design is $ccl_G(H_2)$,
- the block set is $ccl_G(H_1)$,
- the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, ..., G_k\} \subset \{H_2^x \cap H_1^y \mid x, y \in G\}$.
- $\mathcal{D}$ can have repeated blocks,
- The group $G / \bigcap_{K \in ccl_G(H_2) \bigcup ccl_G(H_1)} N_G(K)$ acts as an automorphism group of $\mathcal{D}$.

Tablica: Maximal subgroups, up to conjugation, of the group $S(6,2)$

Maximal subgroup	Structure	Order	Index
M_1	$U(4,2):Z_2$	51840	28
M_2	S_8	40320	36
M_3	$E_{32}:S_6$	23040	63
M_4	$U(3,3):Z_2$	12096	120
M_5	$E_{64}:L(3,2)$	10752	135
M_6	$((E_{16}:Z_2)\times E_4):(S_4\times S_4)$	4608	315
M_7	$S_3\times S_6$	4320	336
M_8	$L(2,8):Z_3$	504	960

Tablica: Block designs constructed from the group $S(6,2)$

Combinatorial structure	The full automorphism group
2-(28,12,11)	$S(6,2)$
2-(28,4,5)	$S(6,2)$
2-(28,10,40)	$S(6,2)$
2-(36,16,12)	$S(6,2)$
2-(36,8,6)	$S(6,2)$
2-(36,12,33)	$S(6,2)$
2-(36,6,8)	$S(6,2)$
2-(63,31,15)	$PGL(6,2)$
2-(28,7,16)	$S(6,2)$
2-(28,10,45)	$S(6,2)$
2-(28,12,66)	$S(6,2)$
2-(36,16,72)	$S(6,2)$
2-(63,31,90)	$PGL(6,2)$

- E.Witt, Die 5−Facht transitiven Gruppen von Mathieu, Abh. Math. Sem. Univ. Hamburg 12 (1938), 256 − 264.
- E.Witt, Uber Steinersche Systeme, Abh. Math. Sem. Univ. Hamburg 12 (1938), 265 − 275.
- W. Haemers, C. Parker, V. Pless and V. D. Tonchev, A design and a code invariant under the simple group Co_3, J.Combin. Theory, Ser. A 62 (1993), 225 − 233.
- D. Held, J. Hrabe de Angelis and M.-O. Pavčević, $PSp_4(3)$ as a symmetric $(36, 15, 6)$- design, Rend. Semin. Mat. Univ. Padova 101 (1999), 95 − 98.
- C. Parker and V.D. Tonchev, Linear Codes and Dounle Transitive Symmetric Design, Linear Algebra Appl. 226 − 228(1995), 237 − 246.
- U. Dempwolff, Primitive Rank 3 Groups on Symmetric Designs, Des. Codes Cryptography 22, No. 2 (2001), 191 − 207.

- D. Crnković and V. Mikulić, Unitals, projective planes and other combinatorial structures constructed from the unitary groups $U(3, q)$, $q = 3, 4, 5, 7$, *Ars Combin.*, to appear.
- J. D. Key and J. Moori, Codes, Designs and Graphs from the Janko Groups J_1 and J_2, *J. Combin. Math. Combin. Comput.* **40** (2002), 143–159.
- The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.9; 2006. (http://www.gap-system.org)
- L. H. Soicher, DESIGN - a GAP package, Version 1.3, 2006. (http://designtheory.org/software/gap_design/)